LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

B.Sc. DEGREE EXAMINATION – **MATHEMATICS**

FOURTH SEMESTER – **APRIL 2023**

MT 4502 – MODERN ALGEBRA

Date: 02-05-2023 Dept. No. [ ] Max. : 100 Marks

Time: 09:00 AM - 12:00 NOON

PART A

Answer ALL the questions **(10 x 2 = 20)**

1. Let R be the set of all numbers. Define $*$ by $x * y = xy + 1$ for all x,y in R. Show that $*$ is commutative but not associative.
2. Illustrate a partially ordered set.
3. Define a cyclic group.
4. Give an example of an abelian group which is not cyclic.
5. Define an automorphism of a group.
6. Let G be the group of non-zero real numbers under multiplication. and $f: G \to G$ be defined by $f(x)=x^2$ for all $x \in G$. Is f a homomorphism of G into G? Justify your answer.
7. Let Z be the ring of integers. Give all the maximal ideals of Z.
8. What is a division ring.
9. Define an integral domain with an example.
10. What is a Gaussian integer?

PART B

Answer any FIVE questions **(5 x 8 = 40)**

11. If G is a group, then prove that
    i) the identity element of G is unique.
    ii) every a $\in$ G has an unique inverse in G.
12. Prove that a non-empty subset H of a group G is a subgroup of G if and only if $HH=H$ and $H=H^{-1}$.
13. Show that the union of two subgroups of G is a subgroup of G if and only if one is contained in the other.
14. If f is a homomorphism of a ring R into a ring R', then prove that the kernel of f is an ideal of R.
15. Show that a subgroup N of a group G is a normal subgroup of G if and only if every left coset of N in G is a right coset of N in G.
16. If f is a homomorphism of a group G into a group G', then prove that
    (i) $f(e) = e'$ where e' *is* the identity element of G'
    (ii) $f(a^{-1}) = [f(a)]^{-1}$ for all $a \in G$.
17. Let R be an Euclidean ring. Then prove that any two elements a and b in R have a greatest common divisor d which can be expressed by $\lambda a + \mu b$ for some λ, μ in R.
18. Show that every finite integral domain is a field.

PART C

Answer Any TWO question **(2 x 20 = 40)**

19. State and prove the fundamental theorem of arithmetic. (20)

20. (a) State and prove Lagrange's theorem.

    (b) Let H be a subgroup of index 2 in a group G. Prove that H is a normal subgroup. (15+5)

21. (a) State and prove the fundamental theorem of homomorphism on groups.

    (b) Let R be a commutative ring with unit element whose only ideals are *(0)* and R itself. Prove that R is a field. (12+8)

22. State and prove unique factorization theorem. (20)

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